SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2005

First Assessment

Mathematics
Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used
- All necessary working should be shown in every question

Total Marks – 60

- All Questions may be attempted
- Each Question is worth 15 marks, and should be handed up in a separate examination Booklet.

Examiner – A.M.Gainford

Question 1. (15 Marks) (**Start a new booklet**.)

Marks

(a) The point $P(6, 9)$ divides the interval AB externally in the ratio 3:2. Find the coordinates of the point B given that A is (1, 4). **2**

(b) The derivative function of a curve is given by $\frac{dy}{dx} = 2x - 3$. If the curve passes through the point (-1, -3), find its equation. **2**

(c) Find $\frac{dy}{dx}$ given: **6**

(i) $y = \sqrt{3x^2 - 4}$

(ii) $y = x^3(2 - x)^4$

(iii) $y = \frac{(x-1)^2}{2x+1}$

(d) Solve the inequalities **5**

(i) $\frac{1}{x} > \frac{1}{5}$

(ii) $\frac{x}{x-1} \le 2$

Question 2. (15 Marks) (**Start a new booklet**.)

Marks

(a) Find a general solution of the equation **2**

$$\sin\theta\cos\theta = \frac{1}{2}.$$

(b) Given the polynomial $P(x) = x^3 + 2x^2 - 11x - 12$: **4**

(i) Use the factor theorem to determine the zeroes of the polynomial.

(ii) Express $P(x)$ as a product of linear factors.

(c) Prove the identity **2**

$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$$

(d) Find the acute angle between the lines $x - 2y + 1 = 0$ and $3x + 4y - 3 = 0$, correct to the nearest minute. **3**

(e) Fred wishes to purchase an annuity which will pay him \$28 000 at the end of each year, for a period of 20 years. The ruling interest rate is $9\frac{1}{2}\%$ per annum compound. How much will Fred have to pay for the annuity? **4**

Question 3. (15 Marks) (**Start a new booklet**.)

Marks

(a) Solve the equation $2x^3 + 3x^2 - 3x - 2 = 0$ given that the roots are in geometric progression. **4**

(b) From a point A on the ground the angle of elevation of the top (T) of a tower is 35°. **5**

(i) Find the angle of elevation of the top of the tower from a point B on the same level as A, but twice as far from the base (X) of the tower.

(ii) If A is due south of the tower and B is 130 metres due west of A, sketch a diagram to represent this situation.

(iii) Find the height of the tower to the nearest metre.

(c) (i) Show that the point $P(4t, 2t^2)$ lies on a parabola for all values of t, and state its focus and directrix. **4**

(ii) Find the chord of contact to the parabola from the point X(2, -2).

(iii) Show that this chord passes through the focus of the parabola.

(d) Use the t-method (where $t = \tan\frac{\theta}{2}$) to solve $\sin\theta + 3\cos\theta = 2$ where $-180^\circ \le \theta \le 180^\circ$. Express your answer correct to the nearest minute. **2**

Question 4. (15 Marks) (**Start a new booklet**.)

Marks

(a) Consider the polynomial $P(x) = x^4 + x^3 - 8x^2 - ax - 9$: **4**

(i) Find the value of the constant a if $P(x)$ is divisible by $Q(x) = x^2 - 9$.

(ii) Show that for this value of a the only real zeroes of $P(x)$ are those of $Q(x)$.

(b) P and Q are points on the parabola with parametric equations $x = 2at,\ y = at^2$, where $t = p$ and $t = q$ respectively. **7**

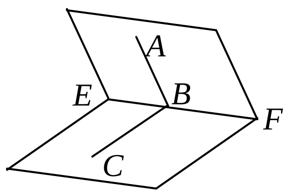
(i) State the coordinates of M, the midpoint of PQ, in terms of a, p and q.

(ii) Show that if the chords PO and QO, where O is the origin, are perpendicular, then $pq = -4$.

(iii) Hence show that the locus of M is a parabola, and state its vertex and focal length.

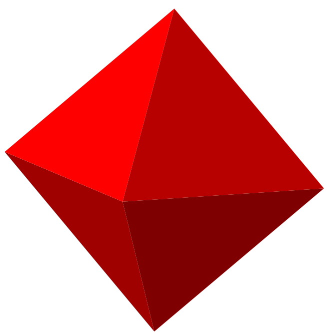
(iv) Given that the equation of PQ is $a(p+q)x = 2a(y + apq)$ find the point of intersection of the tangents at P and Q.

(c) It is given that the angle between two planes is the angle between two lines both perpendicular at the same point to the line of intersection of the planes.



$AB \perp EF$, $BC \perp EF$, and $\angle ABC$ is the angle between the planes.

Find the angle between two adjacent faces of a regular octahedron (in which all edges are of equal length).
Give your answer correct to the nearest minute. **4**



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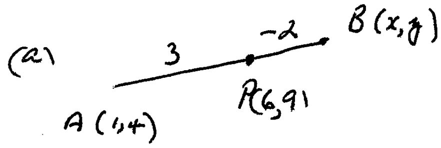
Assessment Task #1

Mathematics Extension 1

Sample Solutions

Question	Marker
1	PRB
2	DMH
3	CK
4	PSP

Question 1

(a) 

$6 = \dfrac{3x-2}{3-2} \Rightarrow 3x = 8 \therefore x = \dfrac{8}{3}$

$9 = \dfrac{3y-8}{3-2} \rightarrow 3y = 17 \therefore y = \dfrac{17}{3}$

$\therefore \boxed{B \text{ is } \left(\frac{8}{3}, \frac{17}{3}\right).}$

(b) N/A.

(c)(i) $y = \sqrt{3x^2-4}$

$= (3x^2-4)^{\frac{1}{2}}.$

$y' = \frac{1}{2}(3x^2-4)^{-\frac{1}{2}} \times 6x.$

$\boxed{y' = \dfrac{3x}{\sqrt{3x^2-4}}}$

(ii) $y = x^3(2-x)^4$

$y' = 3x^2.(2-x)^4 + x^3.4(2-x)^3 \times -1.$

$= x^2(2-x)^3[3(2-x) - 4x]$

$= x^2(2-x)^3(6-3x-4x)$

$\therefore \boxed{y' = x^2(2-x)^3(6-7x).}$

(iii) $y = \dfrac{(x-1)^2}{2x+1}$

$y' = \dfrac{(2x+1).2(x-1)^1 - (x-1)^2.2}{(2x+1)^2}$

$= \dfrac{2(x-1)[2x+1-(x-1)]}{(2x+1)^2}$

$\boxed{y' = \dfrac{2(x-1)(x+2)}{(2x+1)^2}}$

(d)(i) $\dfrac{1}{x} > \dfrac{1}{5}$

$\therefore \dfrac{5}{x} - 1 > 0$

$\dfrac{5-x}{x} > 0$ (× both sides by x^2)

$x(5-x) > 0$

$\therefore \boxed{0 < x < 5}$

(ii) $\dfrac{x}{x-1} \leq 2.$

$\dfrac{x}{x-1} - 2 \leq 0$

$\dfrac{x - 2(x-1)}{x-1} \leq 0.$

$\dfrac{2-x}{x-1} \leq 0$ [× both sides by $(x-1)^2$]

$(2-x)(x-1) \leq 0.$

$\therefore \boxed{x < 1,\ x \geq 2.}$

[NB $x \neq 1$].

* DO NOT WRITE AS $2 \leq x < 1$!!

Question 2

(a) Find the general solution of the equation

$$\sin\theta\cos\theta = \frac{1}{2}.$$

Solution:

$$\begin{aligned} 2\sin\theta\cos\theta &= 1,\\ \sin 2\theta &= 1,\\ 2\theta &= n\pi + (-1)^n\frac{\pi}{2} \text{ where } n \in \mathbf{J},\\ &= \frac{\pi}{2} + 2n\pi,\\ \theta &= n\pi + \frac{\pi}{4}. \end{aligned}$$

(b) Given the polynomial $P(x) = x^3 + 2x^2 - 11x - 12$:

(i) Use the factor theorem to determine the zeroes of the polynomial.

Solution: Factors of 12 are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12.

$$\begin{aligned} P(1) &= 1 + 2 - 11 - 12 \neq 0,\\ P(-1) &= -1 + 2_1 1 - 12 = 0. \end{aligned}$$

$$\begin{array}{rl} & \phantom{x^3+{}} x^2 \phantom{{}-11} + x - 12 \\ \cline{2-2} x+1 \big) & x^3 + 2x^2 - 11x - 12 \\ & -x^3 - x^2 \\ \cline{2-2} & \phantom{-x^3 -{}} x^2 - 11x \\ & - x^2 - x \\ \cline{2-2} & -12x - 12 \\ & 12x + 12 \\ \cline{2-2} & 0 \end{array}$$

$$\text{or} \quad \begin{array}{r|rrrr} & 1 & 2 & -11 & -12 \\ -1 & & -1 & -1 & 12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

Now $x^2 + x - 12 = (x+4)(x-3)$

$\therefore$ zeroes are -1, -4, 3.

(ii) Express $P(x)$ as a product of linear factors.

Solution: $P(x) = (x+1)(x+4)(x-3)$

(c) Prove the identity

$$\frac{\sin\theta}{1-\cos\theta}+\frac{1-\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$$

Solution:
$$\begin{aligned}\text{L.H.S.} &= \frac{\sin^2\theta+1-2\cos\theta+\cos^2\theta}{\sin\theta(1-\cos\theta)},\\ &= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)},\\ &= \frac{2}{\sin\theta},\\ &= 2\operatorname{cosec}\theta,\\ &= \text{R.H.S.}\end{aligned}$$

(d) Find the acute angle between the lines $x-2y+1=0$ and $3x+4y-3=0$, correct to the nearest minute.

Solution: $m_1=\frac{1}{2}$, $\quad m_2=-\frac{3}{4}$.

$$\begin{aligned}\tan\alpha &= \left|\frac{\frac{1}{2}+\frac{3}{4}}{1-\frac{1}{2}\times\frac{3}{4}}\right|,\\ &= \left|\frac{4+6}{8-3}\right|,\\ &= 2.\end{aligned}$$

$\therefore$ Acute angle is $63^\circ 26'$ (nearest minute).

(e) Fred wishes to purchase an annuity which will pay him \$28 000 at the end of each year, for a period of 20 years. The ruling interest rate is $9\frac{1}{2}\%$ per annum compound. How much will Fred have to pay for the annuity?

Solution: Let purchase price be $\$P$.

$$\begin{aligned}\text{Value after 1 year} &= P\times 1{\cdot}095-28\,000,\\ \text{Value after 2 years} &= (P\times 1{\cdot}095-28\,000)1{\cdot}095-28\,000,\\ \text{Value after 3 years} &= ((P\times 1{\cdot}095-28\,000)1{\cdot}095-28\,000)1{\cdot}095-28\,000,\\ &= P\times 1{\cdot}095^3-28\,000(1+1{\cdot}095+1{\cdot}095^2),\\ &= P\times 1{\cdot}095^3-\frac{28\,000(1{\cdot}095^3-1)}{0{\cdot}095},\\ \text{Value after 20 years} &= P\times 1{\cdot}095^{20}-\frac{28\,000(1{\cdot}095^{20}-1)}{0{\cdot}095},\\ &= 0.\\ \textit{i.e.},\ P\times 1{\cdot}095^{20} &= \frac{28\,000(1{\cdot}095^{20}-1)}{0{\cdot}095},\\ \text{So } P &= \frac{28\,000(1{\cdot}095^{20}-1)}{1{\cdot}095^{20}\times 0{\cdot}095},\\ &= 246\,746{\cdot}70 \text{ (2 d.p.)}\end{aligned}$$

$\therefore$ Annuity will cost \$246 746·70.

Question 3

(a) Let roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$

$$\frac{\alpha}{\beta} + \alpha + \alpha\beta = -\frac{3}{2} \quad \text{— (1)}$$

$$\frac{\alpha^2}{\beta} + \alpha^2 + \alpha^2\beta = -\frac{3}{2} \quad \text{— (2)}$$

$$\alpha^3 = 1 \quad \text{— (3)}$$

From (3) $\alpha = 1$

$\Rightarrow$ (1) and (2) [same eq^n]

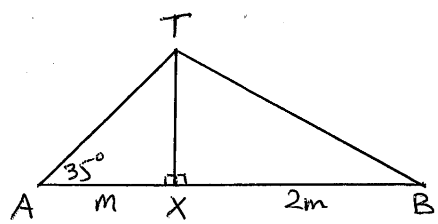
ie $\frac{1}{\beta} + 1 + \beta = -\frac{3}{2}$

$$2\beta^2 + 5\beta + 2 = 0$$

$\therefore \beta = -2, -\frac{1}{2}$

Roots are $1, -2, -\frac{1}{2}$

(b) (i)



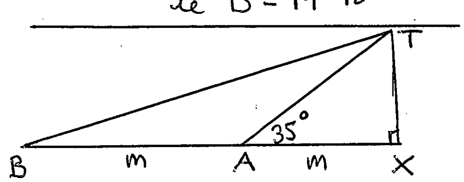
$\tan 35° = \frac{TX}{m}$ and $\tan B = \frac{TX}{2m}$

$\Rightarrow m\tan 35° = 2m\tan B$

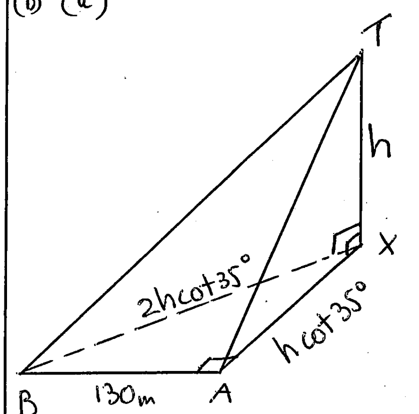
$$\frac{1}{2}\tan 35° = \tan B$$

ie $\hat{B} = 19°18'$

OR



(b) (ii)



(iii) In $\triangle ABX$ by Pyth. Theorem

$$130^2 + (h\cot 35)^2 = (2h\cot 35)^2$$

$$h = \frac{130\tan 35°}{\sqrt{3}} \doteqdot 53\text{m}$$

(c) (i) $x = 4t, \quad y = 2t^2$

$\Rightarrow t = \frac{x}{4} \quad \therefore y = 2\left(\frac{x}{4}\right)^2$

ie $x^2 = 8y$

(ii) FOCUS $(0,2)$ DIR. $y = -2$

Eq^n of c of c is

$$xx_0 = 2a(y + y_0)$$

$$\therefore \quad 2x = 4(y-2)$$

$$x = 2y - 4$$

(iii) Focus $(0,2)$

$\Rightarrow \quad 0 = 2(2) + 4$

pt satisfies eq^n of c of c

$\therefore$ lies on $x = 2y - 4$

(d)

$$\sin\theta + 3\cos\theta = 2$$

$$\frac{2t}{1+t^2} + 3\left[\frac{1-t^2}{1+t^2}\right] = 2$$

$$2t + 3 - 3t^2 = 2 + 2t^2$$

$$5t^2 - 2t - 1 = 0$$

$$t = \frac{1 \pm \sqrt{6}}{5}$$

$$\tan\frac{\theta}{2} = \frac{1 \pm \sqrt{6}}{5}$$

$$\frac{\theta}{2} = 34^\circ 36', \quad -16^\circ 10'$$

$$\theta = 69^\circ 12', \quad -32^\circ 20'$$

NB Clearly $\theta = 180^\circ$ is not a solution of the original equation

Question 4

(a) (i) $P(x) = x^4 + x^3 - 8x^2 - ax - 9$

$= \left(x^2 - 9\right)Q\left(x\right)$

$\therefore P(3) = P\left(-3\right) = 0$ (Factor Theorem)

$P\left(3\right) = 3^4 + 3^3 - 8\left(3\right)^2 - 3a - 9 = 0 \Rightarrow a = 9$

$P\left(-3\right) = \left(-3\right)^4 + \left(-3\right)^3 - 8\left(-3\right)^2 + 3a - 9 = 0 \Rightarrow a = 9$

$\therefore a = 9$

(ii) $P\left(x\right) = \left(x^2 - 9\right)\left(x^2 + Bx + 1\right)$

$P\left(1\right) = -24$

$\therefore -24 = \left(-8\right)\left(2 + B\right) \Rightarrow B = 1$

$\therefore P\left(x\right) = \left(x^2 - 9\right)\left(x^2 + x + 1\right)$

For $P\left(x\right) = 0$ then either

1. $x^2 - 9 = 0 \Rightarrow x = \pm 3$ **OR**
2. $x^2 + x + 1 = 0 \Rightarrow \Delta = -3$

Clearly with $\Delta = -3$ then $x^2 + x + 1 = 0$ has no **REAL** solutions.
So the only real **zeroes** of $P\left(x\right)$ are those of $Q\left(x\right) = x^2 - 9$

(b) $P\left(2ap, ap^2\right), Q\left(2aq, aq^2\right)$

(i) $M\left\{a\left(p + q\right), a\left(\dfrac{p^2 + q^2}{2}\right)\right\}$

(ii) $OP \perp OQ \Rightarrow m_{OP} \times m_{OQ} = -1$

$m_{OP} = \dfrac{ap^2}{2ap} = \dfrac{p}{2}$

$\therefore m_{OQ} = \dfrac{q}{2}$

$\therefore \dfrac{p}{2} \times \dfrac{q}{2} = -1 \Rightarrow pq = -4$

(iii) $x = a(p+q) \Rightarrow \dfrac{x}{a} = p+q$

$$y = a\left(\frac{p^2+q^2}{2}\right) \Rightarrow \frac{2y}{a} = p^2+q^2$$

$$\left(\frac{x}{a}\right)^2 = (p+q)^2 = p^2+q^2+2pq = \frac{2y}{a} - 8$$

$$\therefore x^2 = 2ya - 8a^2 = 2a(y-4a)$$

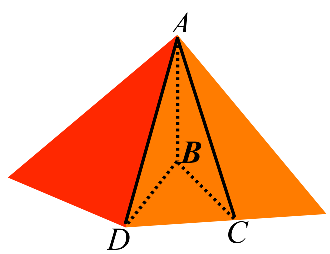
Clearly this is the equation of a parabola with vertex $(0,4a)$ with a focal length of $\frac{1}{2}|a|$

(iv) Clearly PQ is the chord of contact from the point $T(x_0, y_0)$ where $xx_0 = 2a(y+y_0)$ and $T(x_0, y_0)$ is the point of intersection of the tangents from P and Q.

$$\therefore a(p+q)x = 2a(y+apq) \Rightarrow \begin{cases} x_0 = a(p+q) \\ y_0 = apq \end{cases}$$

So T is the point $\{a(p+q), apq\}$ or $\{a(p+q), -4a\}$ $\quad (\because pq = -4)$

(c)



The angle we require is $\theta = 2 \times \angle ACB$.
Let the sides of the octahedron be 2a units.

$\therefore DC = BC = a \Rightarrow DB = a\sqrt{2}$ (Pythagoras' Theorem)

$\therefore AB = \sqrt{4a^2 - 2a^2} = a\sqrt{2}$ (Pythagoras' Theorem)

$$\therefore \tan(\angle ACB) = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\therefore \theta = 2\tan^{-1}\sqrt{2} = 109°\ 28'$$